Connection dynamics of a gauge theory of gravity coupled with matter

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We study the coupling of the gravitational action, which is a linear combination of the Hilbert-Palatini term and the quadratic torsion term, to the action of Dirac fermions. The system possesses local Poincare invariance and hence belongs to Poincare gauge theory with matter. The complete Hamiltonian analysis of the theory is carried out without gauge fixing, which leads to a consistent geometrical dynamics with second-class constraints and torsion. After performing a partial gauge fixing, all second-class constraints can be solved, and a connection dynamical formalism of the theory can be obtained by a canonical transformation. Hence, the techniques of loop quantum gravity can be employed to quantize this Poincare gauge theory with non-zero torsion.

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I. INTRODUCTION

General Relativity(GR) has been very successful in describing universe at large scales. However, it is believed that we have to develop a quantum theory of gravity for a consistent description of nature. One of the reasons that classical GR cannot be consistent can be seen from the Einstein's equations which relate gravitational and matter degrees of freedom. While the gravitational part is classical and is encoded in the Einstein tensor, since matter interactions are very well described by quantum field theory, we need to use some quantum version of the stress energy tensor for the matter part. This would imply that a consistent coupling of matter and gravity for all energy scales requires both of them to be quantized.

Einstein's equations can be obtained via an action principle starting from the first-order Hilbert-Palatini action. However, if we consider fermionic matter sources, the equations of motion from this action will not provide the torsion-free condition of vacuum case. Hence, we have to either allow for torsion or make some suitable modification of the action. (See [1] and references therein for a comprehensive account of torsion in gravity). So, if one wants to start with first-order action, it is very possible that quantum theory of gravity would incorporate torsion in its formalism in order to consistently couple gravity to fermions. Among various attempts to look for a quantum gravity theory, gauge theories of gravity are very attractive since the idea of gauge invariance has already been successful in the description of other fundamental interactions. Local gauge invariance is a key concept in Yang-Mills theory. Together with Poincare symmetry, it lays the foundation of standard model in particle physics. Localization of Poincare symmetry leads to Poincare Gauge Theory(PGT) of gravity. One of the key features in PGT is that, in general, gravity is not only represented as curvature but also as torsion of space-time. GR is a special case of PGT when torsion equals zero.

PGT provides a very convenient framework for studying theories with torsion. A number of actions which satisfy local Poincare symmetry have been analyzed by various researchers ([2] provides a comprehensive review and bibliography of the progress made in PGT). However, one of the drawbacks of PGT is that its Hamiltonian formulation is usually very complicated. Although Hamiltonian analysis is performed for many models in PGT, the results are at a formal level without explicit expressions of the additional required second-class constraints. From the point of view of canonical quantization, it is essential to have a well-defined consistent Hamiltonian theory at the classical level. Such an ingredient is missing if we want to incorporate torsion into candidate quantum gravity models constructed from PGT. Moreover, the internal gauge group in PGT is in general non-compact, while most of the standard tools developed in quantum field theory apply to gauge theories with compact gauge groups.

There exists a well-known $SU(2)$ gauge theory formulation of canonical GR [3, 4], where the basic variables are the

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densitized triad and Ashtekar-Barbero connection. A candidate canonical quantum gravity theory known as Loop Quantum Gravity (LQG) [5–8] can be constructed starting from the connection dynamical formulation. Moreover, LQG can also be extended to some modified gravity theories such as, $f(R)$ theories [9, 10] and scalar-tensor theories [11]. However, the action of GR from which the connection dyanimics can be derived is not the standard Hilbert-Palatini action. An additional term known as the Holst term has to be added to the standard Hilbert-Palatini action in order to rewrite GR as a $SU(2)$ gauge theory[12, 13]. It is customary to multiply the additional Holst term with a coupling constant γ known as the Immirzi parameter. Classically these two actions are equivalent since the additional Holst term does not affect the equations of motion although it is not a total derivative. The parameter γ does not appear in the classical equations of motion. This is because the Holst term differs from a total derivative known as the Nieh-Yan term [14] by a term quadratic in torsion (for the exact relations between them see [15, 16]). Since the torsion term is zero when there is no fermionic matter, the Nieh-Yan term and the Holst term are same, and hence the connection dynamics obtained from adding either term to the Hilbert-Palatini action would be equivalent. It has been shown that a $SU(2)$ gauge theory can also be constructed from an action containing the standard Hilbert-Palatini term and the Nieh-Yan term [15]. However, when there are fermions, the T^2 term is not zero and the the difference in the Holst term and the Nieh-Yan term shows up. In [17] it was found that adding the standard fermion action along with the Holst term leads to equations of motion which depend on γ and are therefore not equivalent to standard GR with fermions. The difference arises because the Holst term is not a total derivative. In [15] it was shown that there is no such issue if the full Nieh-Yan term is used. An alternative possibility of modifying the fermion action to be *non-minimally coupled* has been analyzed in detail in [18, 19]. The additional piece in fermion action cancels the contribution of the Holst piece if the coupling constants are chosen accordingly. (See [20] for a recent account of these issues). In the absence of direct experimental or observational evidence of quantum gravity and of torsion, it is not clear which action should be the appropriate starting point for quantization, particularly from the perspective of LQG. It is therefore very important to study all the different possibilities. However to apply the LQG techniques, it is essential to first reformulate these candidates as gauge theories with a compact gauge group.

In this series of works, instead of the Holst piece of the Nieh-yan term, we consider the T^2 piece. In [21] we considered the vacuum case, i.e. an action with only this T^2 along with the standard Hilbert-Palatini term. Instead of the Immirzi parameter γ, we use an arbitrary coupling constant α which is related to γ by numerical factors. There it was shown that, although we started from an action with explicit torsion dependence, the conditions of the closure of the constraint algebra imply that torsion is zero, and hence we go back to standard GR. This is consistent with the results that there is no torsion in the absence of spinors. However, in that case it was not possible to obtain a $SU(2)$ gauge theory. The variables we choose are motivated by PGT. But unlike other analyses in PGT we obtain explicit expressions of the second-class constraints.

In this paper, we add Dirac fermions to the action and apply the techniques developed in [21] to carry out the Hamiltonian analysis. We consider the fermions to be non-minimally coupled, because the T^2 term is not a total derivative and indeed, by proper choice of the two coefficients, the contribution of the additional non-minimal piece is cancelled by the contribution of the torsion piece in the closure of the constraint algebra. Also the relation between torsion and the fermions we obtain is the same as the one obtained in [15] with Nieh-Yan term and minimally coupled fermion action. To the best of our knowledge, this is the first action with explicit torsion terms which has been reformulated as a Hamiltonian $SU(2)$ gauge theory. The new connection we obtain has some properties which are novel and very different from the other connection dynamics formulations obtained so far. The classical system we obtain in this paper can subsequently be loop quantized using the tools already developed in LQG and will lead to a new and inequivalent quantum theory. Also, Hamiltonian formulation of theories with torsion are usually very complicated. We think that the techniques developed in this and the previous paper[21] can be used for analyzing other similar actions with torsion terms. If that is possible, then the general programme of loop quantization can be applied to a much wider class of theories which include torsion.

The paper is organized as follows. In Section (II) we give the explicit expression of the action we start with and our definitions and conventions. In Section (III) we perform a $3+1$ decomposition of this action and perform the Hamiltonian analysis without fixing time gauge. Having obtained a consistent Hamiltonian system, we fix time gauge and then solve the second class constraints in Section (IV). Fixing the time gauge also breaks the $SO(3,1)$ gauge invariance to $SU(2)$. Then in Section (V) we define a new connection which is conjugate to the densitized triad to obtain a $SU(2)$ gauge theory. Our analysis has several novel and peculiar features. We conclude with a discussion of these and some comparison of our results with those obtained using the Holst and Nieh-Yan terms in Section (VI). We will restrict ourselves to 4 dimensions. The Greek letters $\mu, \nu \ldots$ refer to space-time indices while the uppercase Latin letters $I, J \ldots$ refer to the internal $SO(3,1)$ indices. Our spacetime metric signature is $(- + + +)$. Later when we do the $3+1$ decomposition of spacetime, we will use the lowercase Latin letters from the beginning of the alphabet $a, b, \ldots$ to represent the spatial indices. After we reduce the symmetry group to $SU(2)$, the internal indices will be represented by lowercase Latin letters from the middle of the alphabet $i, j \ldots$.

II. THE ACTION

In this paper we consider an action which has three pieces, a Hilbert-Palatini term, a term quadratic in torsion and a term for the massless fermionic matter. It reads

$$S = S_{HP} + \alpha S_T + S_M \quad , \tag{1}$$

where

$$\begin{aligned}
S_{HP} &= \int \mathrm{d}^4x\ eR = \int \mathrm{d}^4x e e^\mu_I e^\nu_J R_{\mu\nu}{}^{IJ}(\omega_\mu{}^{IJ}) \quad , \\
S_T &= \frac{1}{8}\int \mathrm{d}^4x \epsilon^{\mu\nu\rho\sigma} T^I_{\ \mu\nu} T_{I\rho\sigma} \quad , \\
S_M &= ie \int \mathrm{d}^4x \left[\overline{\lambda}(1+i\beta\gamma^5)\gamma^\mu D_\mu\lambda - \overline{D_\mu\lambda}\gamma^\mu(1+i\beta\gamma^5)\lambda\right] \quad .
\end{aligned}$$

Here e^μ_I is the tetrad, e denotes the absolute value of the determinant of the co-tetrad, $\omega_\mu{}^{IJ}$ is the spacetime spin-connection which is not torsion-free, $\epsilon^{\mu\nu\rho\sigma}$ denotes the 4-dimensional Levi-Civita tensor density, and the covariant derivatives in the fermion action read,

$$D_\mu\lambda = \partial_\mu\lambda + \frac{1}{2}\ \omega^{IJ}_\mu \sigma_{IJ}\ \lambda \quad ; \quad \overline{D_\mu\lambda} = \partial_\mu\overline{\lambda} - \frac{1}{2}\ \overline{\lambda}\ \omega^{IJ}_\mu \sigma_{IJ}.$$

Note that we denote $\gamma^\mu = \gamma^I e^\mu_I$ with 4-dimensional Dirac matrices γ^I , $\sigma_{IJ} := \frac{1}{4}[\gamma_I, \gamma_J]$ and $\gamma_5 := i\gamma_0\gamma_1\gamma_2\gamma_3$. Our conventions regarding the Dirac matrices and their properties are given in Appendix (A). Note also that λ and $\overline{\lambda} := \lambda^\dagger\gamma^0$, representing the fermionic degrees of freedom, are 4-dimensional row and column vector respectively. Further,

$$\begin{aligned}
R_{\mu\nu}{}^{IJ} &= \partial_{[\mu}\omega_{\nu]}{}^{IJ} + \omega_{[\mu}{}^{IK}\omega_{\nu]K}{}^{J}, \qquad (2)\\
T^I_{\ \mu\nu} &= \partial_{[\mu}e^I_{\nu]} + \omega_{[\mu}{}^{I}{}_{|J|}e^J_{\nu]} \qquad (3)
\end{aligned}$$

are the definitions for curvature and torsion respectively. [1] The action (1) is invariant under local Poincare transformations. We will be working in the first-order formalism and hence both the co-tetrad e^I_μ and the spin connection $\omega_\mu{}^{IJ}$ are treated as independent fields. Our covariant derivative D_μ acts in the following way:

$$D_\mu e^I_\nu := \partial_\mu e^I_\nu + \omega_\mu{}^I{}_J e^J_\nu.$$

Note that the coupling parameter α in action (1) is a non-zero real number. The parameter β in the matter action denotes nonminimal coupling and with $\beta = 0$ we get back minimally coupled Fermion action. The parameter β, in general, has no relation with the parameter α. However in the subsequent Hamiltonian analysis, we have to choose $\beta = \frac{\alpha}{2}$ for the closure of the constraint algebra. We therefore adopt that relation between the two parameters from here onwards.

By variation of the action we can find the equations of motion for the various fields, namely e^I_μ, $\omega_\mu{}^{IJ}$, λ and $\overline{\lambda}$. In particular the equation of motion for the connection $\omega_\mu{}^{IJ}$ is given by

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}e^K_\nu D_{[\rho}e^L_{\sigma]}\left[\frac{\alpha}{2}(\eta_{JK}\eta_{IL} - \eta_{IK}\eta_{JL}) - \epsilon_{IJKL}\right] - \frac{1}{2}ee^\mu_K\overline{\lambda}\gamma_5\gamma_L\lambda\left[\frac{\alpha}{2}(\eta_{IK}\eta_{JL} - \eta_{JK}\eta_{IL}) + \epsilon_{IJKL}\right] = 0. \tag{4}$$

It is easy to see that, as expected in the presence of spinors, the connection is not torsion free, i.e., the equation of motion do not imply $D_{[\rho}e^L_{\sigma]} = 0$. In [21], the Hamiltonian analysis of the action (1) without the matter part was carried out. In that case, the Lagrangian equations of motion showed that torsion was zero on-shell although the action has explicit torsion terms. In the next section we will carry out a similar analysis with action (1) where the torsion is expected to be non-zero.

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[1] Our conventions of symmetrization and antisymmetrization are $A^{(ab)} := A^{ab} + A^{ba}$ and $A^{[ab]} := A^{ab} - A^{ba}$ respectively

III. HAMILTONIAN ANALYSIS

We shall perform the Hamiltonian analysis similar to what was done in [21] for the action without the matter term. Recall that in the Hamiltonian formulation of Hilbert-Palatini theory the basic variables are the $SO(3,1)$ spin connection $\omega_a{}^{IJ}$ and its conjugate momentum. It is well known that this formulation contains second-class constraints. Since our action (1) contains the other term which explicitly depends on torsion, we expect that there will be another pair of conjugate variables and the second-class constraints will be somehow different from the Hilbert-Palatini case. It is well known that in the absence of fermionic matter, torsion is zero. In the analysis of [21], this was obtained as a condition for the closure of the constraint algebra. Owing to the presence of the fermion term in the action, here torsion will not be zero. However the conditions for closure of constraint algebra will determine how the torsion and the spinorial degrees of freedom are related.

A. 3+1 Decomposition

To seek a complete Hamiltonian analysis, we perform the $3+1$ decomposition of our fields without breaking the internal $SO(3,1)$ symmetry and also without fixing any gauge. To identify our configuration and momentum variables for performing Hamiltonian analysis, we can rewrite the three pieces in the action as:

$$S_{HP} = \int \mathrm{d}^4x \Big[e e^t_{[I} e^a_{J]} \left(\partial_t \omega_a{}^{IJ}\right) + e e^t_{[I} e^a_{J]} \left(-\partial_a \omega_t{}^{IJ} + \omega_{[t}{}^{IK} \omega_{a]}{}^{KJ}\right) + \frac{1}{2} e e^a_{[I} e^b_{J]} R_{ab}{}^{IJ} \Big], \tag{5}$$

$$\alpha S_T = \alpha \int \mathrm{d}^4x \Big[\epsilon^{abc} D_b e^I_c \left(\partial_t e^I_a\right) + \epsilon^{abc} D_b e^I_c \left(-\partial_a e^I_t + \omega_{[t}{}^{IJ} e_{a]J}\right) \Big], \tag{6}$$

$$\begin{aligned} S_M = \int \mathrm{d}^4x \; ie \Big[& \left(\overline{\lambda}(1+i\frac{\alpha}{2}\gamma^5)\gamma^t \partial_t \lambda - \partial_t \bar{\lambda}\gamma^t (1+i\frac{\alpha}{2}\gamma^5)\lambda\right) + \frac{1}{2}\left(\overline{\lambda}(1+i\frac{\alpha}{2}\gamma^5)\gamma^t \omega_t^{IJ}\sigma_{IJ}\lambda + \overline{\lambda}\omega_t^{IJ}\sigma_{IJ}\gamma^t(1+i\frac{\alpha}{2}\gamma^5)\lambda\right) \\ & + \left(\overline{\lambda}(1+i\frac{\alpha}{2}\gamma^5)\gamma^a D_a\lambda - \overline{D_a\lambda}\gamma^a(1+i\frac{\alpha}{2}\gamma^5)\lambda\right)\Big]. \end{aligned} \tag{7}$$

We can read off the momenta with respect to $\omega_a{}^{IJ}$, e^I_a, λ and $\overline{\lambda}$ respectively as

$$\Pi^a_{IJ} := e e^t_{[I} e^a_{J]} \quad , \quad \Pi^a_I := \alpha \epsilon^{abc} D_b e_{cI}, \tag{8}$$

$$\overline{\Pi} := ie\overline{\lambda}(1+i\frac{\alpha}{2}\gamma^5)e^t_I\gamma^I \quad , \quad \Pi := -iee^t_I\gamma^I(1+i\frac{\alpha}{2}\gamma^5)\lambda, \tag{9}$$

where ϵ^{abc} denotes the 3-dimensional Levi-Civita tensor density, and we have used the relation $\gamma^\mu = \gamma^I e^\mu_I$. For our analysis we shall use a standard parametrization of the tetrad and the co-tetrad fields as in [22]. This is the same parametrization used in the Hamiltonian analysis of the first two terms of our action in [21]. The details of the parametrization and some useful identities are given in Appendix (B).

After some manipulation and neglecting the total derivatives, the pieces of the action can be written in this parametrization respectively as

$$S_{HP} = \int \mathrm{d}^4x \Big[\Pi^a_{IJ}\partial_t \omega_a{}^{IJ} - \left(\frac{N^2}{2e}\Pi^{[a}_{IK}\Pi^{b]}_{JL}\eta^{KL}R_{ab}{}^{IJ} + \frac{1}{2}N^{[a}\Pi^{b]}_{IJ}R_{ab}{}^{IJ} - \omega_t{}^{IJ}D_a\Pi^a_{IJ}\right)\Big], \tag{10}$$

$$\alpha S_T = \int \mathrm{d}^4x \Big[\Pi^a_I \partial_t V^I_a + \left(NN^I D_a \Pi^a_I + N^a V^I_a D_b \Pi^b_I + \frac{1}{2}\omega_t{}^{IJ}\Pi^a_{[I}V_{J]a}\right)\Big], \tag{11}$$

$$\begin{aligned} S_M = \int \mathrm{d}^4x \Big[& \overline{\Pi}\partial_t\lambda + \Pi\partial_t\bar{\lambda} - \Big(\frac{N}{\sqrt{q}}\Pi^a_{IJ}\left(\overline{\Pi}\sigma^{IJ}D_a\lambda - \overline{D_a\lambda}\sigma^{IJ}\Pi\right) \\ & + N^a\left(\overline{\Pi}D_a\lambda + \overline{D_a\lambda}\Pi\right) + \frac{1}{2}\omega_t{}^{IJ}\left(\overline{\lambda}\sigma_{IJ}\Pi - \overline{\Pi}\sigma_{IJ}\lambda\right)\Big)\Big]. \end{aligned} \tag{12}$$

Collecting them together we can rewrite the total action in the ADM form

$$S_{HP} + \alpha S_T + S_M = \int \mathrm{d}^4x \Big[\Pi^a_{IJ}\partial_t\omega_a{}^{IJ} + \Pi^a_I\partial_t V^I_a + \overline{\Pi}\partial_t\lambda + \Pi\partial_t\bar{\lambda} - \left(NH + N^aH_a + \omega_t{}^{IJ}\mathcal{G}_{tIJ}\right)\Big], \tag{13}$$

where the Hamiltonian, Diffeomorphism and the Gauss Constraints respectively read

$$
\begin{aligned}
H &= \frac{1}{\sqrt{q}}\Pi^a_{IK}\Pi^b_{JL}\eta^{KL}R_{ab}{}^{IJ} - N^I D_a\Pi^a_I + \frac{1}{\sqrt{q}}\Pi^a_{IJ}\left(\overline{\Pi}\sigma^{IJ}D_a\lambda - \overline{D_a\lambda}\sigma^{IJ}\Pi\right), && (14)\\
H_a &= \Pi^b_{IJ}R_{ab}{}^{IJ} - V^I_a D_b\Pi^b_I + \overline{\Pi}D_a\lambda + \overline{D_a\lambda}\Pi, && (15)\\
\mathcal{G}_{tIJ} &= -D_a\Pi^a_{IJ} - \frac{1}{2}\Pi^a_{[I}V_{J]a} + \frac{1}{2}\left(\overline{\lambda}\sigma_{IJ}\Pi - \overline{\Pi}\sigma_{IJ}\lambda\right). && (16)
\end{aligned}
$$

Subsequently we will drop the subscript t from $\mathcal{G}_{tIJ}$ and denote it as $\mathcal{G}_{IJ}$. From the definition of σ_{IJ} and using the properties of gamma matrices (A2), $\mathcal{G}_{IJ}$ can also be written as

$$
\mathcal{G}_{IJ} = D_a\Pi^a_{IJ} + \frac{1}{2}\Pi^a_{[I}V_{J]a} + \frac{\sqrt{q}}{4}\left(\alpha\overline{\lambda}\gamma_5 N_{[I}\gamma_{J]}\lambda + \epsilon_{IJKL}\overline{\lambda}\gamma_5 N^{[K}\gamma^{L]}\lambda\right) \approx 0. \tag{17}
$$

This is just a rewriting of the Gauss constraint in a form which will be useful in some later calculations. We have identified the basic variables in our theory. The gravitational degrees of freedom are incorporated in the pair $\left(\Pi^a{}_{IJ}, \omega_a{}^{IJ}\right)$, which have 36 degrees of freedom and in the pair $\left(\Pi^a_I, V^I_a\right)$, which have 24. The matter degrees of freedom are in the two pairs $\left(\overline{\Pi}, \lambda\right)$ and $\left(\Pi, \overline{\lambda}\right)$ each with 8 degrees of freedom. The total number of degrees of freedom without considering the constraints is therefore 76.

B. Primary Constraints

Let us now consider the constraints in the theory. At this stage we have the following constraints. (i) Since there is no momentum corresponding to $\omega_t{}^{IJ}$, we have to impose 6 primary constraints $\Pi^t_{IJ} \approx 0$. This leads to 6 secondary constraints $\mathcal{G}_{IJ} \approx 0$. (ii) Also there is no momentum corresponding to e^I_t. We have to impose 4 constraints $\Pi^t_I \approx 0$ which lead to 3 secondary constraints $H_a \approx 0$ and 1 more secondary constraint $H \approx 0$. (iii) From Eq. (8), we can get two other sets of primary constraints

$$
\begin{aligned}
C^a_I &:= \Pi^a_I - \alpha\epsilon^{abc}D_b V_{cI} \approx 0, && (18)\\
\Phi^a_{IJ} &:= \Pi^a_{IJ} - \frac{1}{2}\epsilon^{abc}\epsilon_{IJKL}V^K_b V^L_c \approx 0. && (19)
\end{aligned}
$$

From (18) we get 12 constraints, while (19) gives 18 because of the antisymmetry in IJ. (iv) From the definition of the momenta corresponding to the fermions (Eq. (9)) we get 8 further constraints

$$
\begin{aligned}
\Psi &:= \Pi - i\sqrt{q}N_K\gamma^K\left(1 + i\frac{\alpha}{2}\gamma^5\right)\lambda \approx 0,\\
\overline{\Psi} &:= \overline{\Pi} + i\sqrt{q}\overline{\lambda}\left(1 + i\frac{\alpha}{2}\gamma^5\right)N_K\gamma^K \approx 0. && (20)
\end{aligned}
$$

These are the primary constraints of our theory.

Including all of above constraints we can write the total Hamiltonian as

$$
H_T := NH + N^a\tilde{H}_a + \Lambda^{IJ}\mathcal{G}_{IJ} + \gamma^I_a C^a_I + \lambda^{IJ}_a\Phi^a_{IJ} + u\Psi + \overline{u}\overline{\Psi} \tag{21}
$$

where the expressions of the constraints are given in equations (14), (15), (16), (18), (19) and (20) respectively and the functions multiplying the constraints are the Lagrangian multipliers. At this point they are completely arbitrary. We now need to check whether the Hamiltonian system is consistent. To ensure the consistency of the Hamiltonian system, the constraints have to be preserved under evolution, i.e., the constraint algebra is closed.

Note that the Gauss constraint $\mathcal{G}_{IJ}$ generates the $SO(3,1)$ transformations and is first class. However, as in [21] the constraint which actually generates the spatial diffeomorphisms for the gravitational degrees of freedom is a combination given by

$$
\tilde{H}_a := H_a + \omega_a{}^{IJ}\mathcal{G}_{IJ} + \frac{1}{\alpha}\epsilon_{abc}C^b_I\Pi^c_I \tag{22}
$$

This can be easily demonstrated as:

$$
\delta^{\tilde{H}_a}\omega_c{}^{IJ} := \left\{\omega_c{}^{IJ}, \tilde{H}_a(N^a)\right\} = N^a\partial_a\omega_c{}^{IJ} + \omega_a{}^{IJ}\partial_c N^a = \mathcal{L}_{N^a}\omega_c{}^{IJ},
$$

$$\begin{aligned}
\delta^{\tilde{H}_a}\Pi^c_{IJ} &:= \left\{\Pi^c_{IJ}, \tilde{H}_a(N^a)\right\} = N^a\partial_a\Pi^c_{IJ} - \Pi^a_{IJ}\partial_a N^c + \Pi^c_{IJ}\partial_a N^a = \mathcal{L}_{N^a}\Pi^c_{IJ}, \\
\delta^{\tilde{H}_a}V^I_c &:= \left\{V^I_c, \tilde{H}_a(N^a)\right\} = N^a\partial_a V^I_c + V^I_a\partial_c N^a = \mathcal{L}_{N^a}V^I_c, \\
\delta^{\tilde{H}_a}\Pi^c_I &:= \left\{\Pi^c_I, \tilde{H}_a(N^a)\right\} = N^a\partial_a\Pi^c_I - \Pi^a_I\partial_a N^c + \Pi^c_I\partial_a N^a = \mathcal{L}_{N^a}\Pi^c_I.
\end{aligned} \tag{23}$$

For the matter degrees of freedom the constraint (22) acts as

$$\begin{aligned}
\delta^{\tilde{H}_a}\lambda = \left\{\lambda, \tilde{H}_a(N^a)\right\} = N^a\partial_a\lambda \quad , \quad \delta^{\tilde{H}_a}\overline{\Pi} = \left\{\overline{\Pi}, \tilde{H}_a(N^a)\right\} = N^a\partial_a\overline{\Pi} + \overline{\Pi}\ \partial_a N^a, \\
\delta^{\tilde{H}_a}\overline{\lambda} = \left\{\overline{\lambda}, \tilde{H}_a(N^a)\right\} = N^a\partial_a\overline{\lambda} \quad , \quad \delta^{\tilde{H}_a}\Pi = \left\{\Pi, \tilde{H}_a(N^a)\right\} = N^a\partial_a\Pi + \Pi\ \partial_a N^a.
\end{aligned} \tag{24}$$

Clearly this combination $\tilde{H}_a$, acting on all the variables, generates Lie derivatives [20] and can therefore be identified as the diffeomorphism constraint. Using the property of Lie derivatives (or by explicit calculation) it can be shown that $\tilde{H}_a$ is first class.

C. Second-Class Constraints and Dirac Brackets

The terms in the constraint algebra which are not weakly zero are respectively

$$\left\{\Phi^a_{IJ}(\lambda^{IJ}_a), H(N)\right\} = \left(\frac{NN_I\Pi^a_J}{\alpha} - \frac{\sqrt{q}}{2}N\overline{\lambda}\gamma_5 V^a_I\gamma_J\lambda\right)\left(\alpha\lambda^{IJ}_a + \epsilon_{IJKL}\lambda^{KL}_a\right), \tag{25}$$

$$\left\{C^a_I(\gamma^I_a), \Phi^b_{JK}(\lambda^{JK}_b)\right\} = \epsilon^{abc}\gamma^I_b V^J_c\left(\alpha\lambda^{IJ}_a + \epsilon_{IJKL}\lambda^{KL}_a\right), \tag{26}$$

$$\left\{C^a_I(\gamma^I_a), H(N)\right\} = -\frac{\alpha N}{\sqrt{q}}\epsilon^{abc}\gamma^I_b V^J_c\left(\overline{D_a\lambda}\ \sigma_{IJ}\Pi - \overline{\Pi}\ \sigma_{IJ}D_a\lambda\right), \tag{27}$$

$$\left\{C^a_I(\gamma^I_a), \overline{\Psi}(\overline{u})\right\} = -i\overline{u}\ \overline{\lambda}\left(1 + i\frac{\alpha}{2}\gamma^5\right)\gamma^J\gamma^I_a\ \Pi^a_{IJ}, \tag{28}$$

$$\left\{C^a_I(\gamma^I_a), \Psi(u)\right\} = iu\ \gamma^J\left(1 + i\frac{\alpha}{2}\gamma^5\right)\lambda\gamma^I_a\ \Pi^a_{IJ}, \tag{29}$$

$$\left\{\overline{\Psi}(\overline{u}), \Psi(u)\right\} = 2i\overline{u}u\sqrt{q}\gamma^I N_I. \tag{30}$$

As mentioned above $\mathcal{G}_{IJ}$ and $\tilde{H}_a$ are first-class constraints. We would also like the Hamiltonian constraint H to be first class. The remaining constraints are second class. Second-class constraints are problematic because the flows generated by them do not lie on the constraint surface. There are two ways of dealing with second-class constraints: First, after obtaining a consistent Hamiltonian system we can solve them and eliminate spurious degrees of freedom. Second, we can use Dirac Brackets instead of Poisson Brackets to define evolution and set the second-class constraints identically to be zero [23].

In this paper we shall construct Dirac Brackets only for the constraints Ψ and $\overline{\Psi}$, while the remaining second-class constraints will be solved after obtaining a consistent Hamiltonian system. If we strictly followed the Dirac procedure, we would have to construct Dirac Brackets using all the second-class constraints. The algebra then became very cumbersome and complicated. On the other hand, Dirac Brackets for only Ψ and $\overline{\Psi}$ are easy to construct and, as we shall see later, greatly simplify the calculation. Recall that, for a set of second-class constraints $\{X_m\}$, the Dirac Bracket between two phase space functions f and g is given by

$$\{f, g\}^\star := \{f, g\} - \{f, X_m\}\, M^{-1}_{mn}\, \{X_n, g\} \tag{31}$$

where the matrix M_{mn} given by $M_{mn} = \{X_m, X_n\}$ has a non-zero determinant. For the set of constraints Ψ and $\overline{\Psi}$, the 2×2 matrix M can be easily constructed from Eq (30). Subsequently, we shall use Dirac Brackets instead of Poisson Brackets and set the constraints Ψ and $\overline{\Psi}$ strongly equal to zero. Note that from definition (31) it is obvious that the Dirac Bracket is as same as Poisson Bracket if either f or g commutes with Ψ and $\overline{\Psi}$.

Since the two constraints Ψ and $\overline{\Psi}$ have been eliminated, the total Hamiltonian (21) becomes

$$H_T := NH + N^a\tilde{H}_a + \Lambda^{IJ}\mathcal{G}_{IJ} + \gamma^I_a C^a_I + \lambda^{IJ}_a\Phi^a_{IJ}. \tag{32}$$

Note that the evolutions generated by Poisson Brackets and Dirac Brackets differ only because of the $\gamma^I_a C^a_I$ in H_T, since all other terms in H_T commute with Ψ and $\overline{\Psi}$. Also note that the gauge transformation and diffeomorphism generating characters of $\mathcal{G}_{IJ}$ and $\tilde{H}_a$ respectively do not change when we use Dirac Brackets, since they commute with Ψ and $\overline{\Psi}$.

D. Secondary Constraints

For a consistent Hamiltonian system, the constraints should be preserved under evolution, i.e., for all the constraints C_m, we require $\dot{C_m} := \{C_m, H_T\}^\star \approx 0$. Our analysis will be along the lines of [21]. But owing to presence of fermions, it will turn out that torsion is not zero. As a consequence, the calculations are much more complicated, and the conditions of the closure of constraint algebra will now relate the torsion and the spin degrees of freedom.

Let us first consider the constraint Φ^a_{IJ}. Note that, since Φ^a_{IJ} commutes with Ψ and $\overline{\Psi}$, in this case, its Dirac brackets are as same as its Poisson Brackets. Hence we have

$$\begin{aligned}\dot{\Phi}^a_{IJ}(\sigma^{IJ}_a) := \left\{\Phi^a_{IJ}(\sigma^{IJ}_a), H_T\right\}^\star &= \left\{\Phi^a_{IJ}(\sigma^{IJ}_a), H_T\right\} \\ &= \left\{\Phi^a_{IJ}(\sigma^{IJ}_a), H(N)\right\} + \left\{\Phi^a_{IJ}(\sigma^{IJ}_a), C^b_I(\gamma^I_b)\right\} \approx 0 \end{aligned} \tag{33}$$

where σ^{IJ}_a is an arbitrary smearing function. Using Eqs (25) and (26), and after some calculation, Eq.(33) implies

$$-\epsilon_{abc}\epsilon^{ade}\gamma^{[I}_d V^{J]}_e + \epsilon_{abc}\left(\frac{NN^{[I}\Pi^{aJ]}}{\alpha} - \frac{\sqrt{q}}{2}N\overline{\lambda}\gamma_5 V^{a[I}\gamma^{J]}\lambda\right) \approx 0.$$

Hence we have

$$\left(\gamma^I_b V^J_c - \gamma^I_c V^J_b - \gamma^J_b V^I_c + \gamma^J_c V^I_b\right) - \epsilon_{abc}\frac{N}{\alpha}\left(N^I\Pi^{aJ} - N^J\Pi^{aI} - \frac{\alpha\sqrt{q}}{2}\overline{\lambda}\gamma_5\left[V^{aI}\gamma^J - V^{aJ}\gamma^I\right]\lambda\right) \approx 0. \tag{34}$$

Multiplying (34) with V^b_J and using the properties (B3) we get

$$2\gamma^I_c + V^b_J\gamma^J_b V^I_c - V^b_J\gamma^J_c V^I_b + \epsilon_{abc}\frac{N}{\alpha}N^I\Pi^{aJ}V^b_J - \epsilon_{abc}\frac{N\sqrt{q}}{2}\overline{\lambda}\gamma_5\left(V^{aI}V^b_J\gamma^J - V^{aJ}V^b_J\gamma^I\right)\lambda \approx 0. \tag{35}$$

By multiplying this equation with N_I, V^c_I and V^I_d respectively and using the relations (B2) and (B3), we obtain the following relations

$$\gamma^I_c N_I = \frac{N}{2\alpha}\epsilon_{abc}\Pi^a_J V^b_J, \tag{36}$$

$$\gamma^I_c V^c_I = 0, \tag{37}$$

$$\gamma^I_c V_{dI} = \epsilon_{dbc}\frac{N\sqrt{q}}{2}\overline{\lambda}\gamma_5 V^b_I\gamma^I\lambda, \tag{38}$$

where we have used Eq.(37) to obtain Eq. (38). Finally from the equations (36) and (38) we get a solution for the Lagrangian multiplier γ^I_c as

$$\gamma^I_c = \epsilon_{abc}\frac{N\sqrt{q}}{2}\overline{\lambda}\gamma_5 V^a_I V^b_J\gamma^J\lambda - \frac{N}{2\alpha}\epsilon_{abc}N^I\Pi^a_J V^b_J. \tag{39}$$

Note that, all these equations differ from the corresponding equations in [21] only by the fermion-dependent terms.

So, we have obtained 12 components of γ^I_a from the 18 equations in (34). Consequently there are 6 constraints remaining. By inserting the solutions (39) back into Eq.(34) and after some calculation, we get the following secondary constraint:

$$\chi^{ab} := \Pi^a_I V^b_I + \Pi^b_I V^a_I - \alpha\sqrt{q}\overline{\lambda}\gamma_5 V^a_I V^b_I N_K\gamma^K\lambda \approx 0. \tag{40}$$

Since χ^{ab} is symmetric in $(a \leftrightarrow b)$, it contains just the 6 required constraints. Before proceeding further, let us count the degrees of freedom. As mentioned above, the total number of degrees of freedom without considering the constraints is 76. Now let us count the degrees of freedom removed by the constraints. Clearly $\mathcal{G}_{IJ}$ and $\tilde{H}_a$ are first class. We would like H to also be first class which would give us $(6+3+1) = 10$ first-class constraints removing 20 degrees of freedom. The constraints C^a_I and Φ^a_{IJ} are primary second class removing $12 + 18 = 30$ degrees of freedom. By setting Ψ and $\overline{\Psi}$ strongly equal to zero we have removed 8 more degrees of freedom. Finally the secondary constraint χ^{ab} turns out to be second class and thus removes 6 degrees of freedom. Thus the number of independent degrees of freedom in our system is 12.

E. Relations and Simplifications

We have found all the constraints in our theory. We now need to show that the Hamiltonian system is consistent, i.e., the constraints are preserved under evolution by H_T, where the evolution is defined by using the Dirac Brackets which we have constructed. This will put restrictions on the Lagrange multipliers. We shall solve the Lagrange multipliers of the second-class constraints and keep those of $\mathcal{G}_{IJ}$, $\tilde{H}_a$ and H free. As seen above, $\dot{\Phi}^a_{IJ} \approx 0$ fixed the Lagrange multipliers γ^I_a of the constraint C^a_I to the form given by Eq. (39). This can however be further simplified. For this and for subsequent calculations, we now derive some useful identities using the constraint equations. All these identities hold weakly, i.e., they are true only when the constraints are used.

From the constraints (18) and (19) we can easily obtain the relation:

$$D_{IJ} \;:=\; D_a \Pi^a_{IJ} - \frac{1}{\alpha}\epsilon_{IJKL}\Pi^{aK}V^L_a \;\approx 0\;. \tag{41}$$

Using this and the Gaussian constraint (17) we get, after some algebra,

$$\Pi^a_{[I}V_{J]a} + \frac{\alpha\sqrt{q}}{2}\overline{\lambda}\gamma_5 N_{[I}\gamma_{J]}\lambda \;\approx\; 0. \tag{42}$$

Multiplying this equation with N_J and then with V^b_I and using the properties (B2) and (B3), we get

$$\Pi^b_J N^J \;\approx\; \frac{\alpha\sqrt{q}}{2}\overline{\lambda}\gamma_5\gamma_J V^{bJ}\lambda. \tag{43}$$

By multiplying relation (42) with V^b_I and then with V^c_J and again using the properties (B2) and (B3), we get

$$\Pi^c_I V^b_I - \Pi^b_I V^c_I \;\approx\; 0. \tag{44}$$

Using Eq. (44) in the constraint (40) we get the relation

$$\Pi^a_I V^b_I \;\approx\; \frac{\alpha\sqrt{q}}{2}\overline{\lambda}\gamma_5 V^a_I V^b_I N_K \gamma^K\lambda. \tag{45}$$

These identities can be used to greatly simplify the subsequent calculations.

First, note that because of the identity (44), the second term on the RHS of Eq. (39) drops out and the Lagrangian multiplier of C^c_I in H_T becomes

$$\gamma^I_c \;=\; \epsilon_{abc}\frac{N\sqrt{q}}{2}\overline{\lambda}\gamma_5 V^a_I V^b_J\gamma^J\lambda. \tag{46}$$

This leads to further simplification of our problem. As mentioned above, the difference in evolutions generated by Poisson and Dirac Brackets comes only from the $\gamma^I_a C^a_I$ in H_T, where γ^I_a is an arbitrary Lagrangian multiplier. The condition that Φ^a_{IJ} be preserved under evolution has fixed γ^I_a to the specific form given by Eq. (46). Now recall from Eq. (28), for an arbitrary smearing function η^I_a we have

$$\left\{C^a_I(\eta^I_a), \overline{\Psi}(\overline{u})\right\} = -i\overline{u}\;\overline{\lambda}\left(1 + i\frac{\alpha}{2}\gamma^5\right)\gamma^J\eta^I_a\;\Pi^a_{IJ}.$$

When $\eta^I_a = \gamma^I_a$, which is of the form given in Eq. (46), using (B5) and (B2) we get

$$\gamma^I_c\;\Pi^c_{IJ} \;=\; \left(\epsilon_{abc}\frac{Nq}{2}\overline{\lambda}\gamma_5\gamma^K\lambda\right)V^a_I V^b_K V^c_{[I}N_{J]} = 0.$$

Using this we can see that $\left\{C^a_I(\gamma^I_a), \overline{\Psi}(\overline{u})\right\} \approx 0$. Similarly, we also have $\left\{C^a_I(\gamma^I_a), \Psi(u)\right\} \approx 0$. Therefore, once the Lagrange multiplier γ^I_a is fixed to the value required for a consistent Hamiltonian system, we can calculate the evolution using Poisson Brackets, i.e., for any function f on the phase space we have

$$\dot{f} \;=\; \{f, H_T\}^\star \;\approx\; \{f, H_T\}\,. \tag{47}$$

This key simplification further justifies our choice of Dirac Brackets.

Second, consider the identity (45) again. Multiplying it by V_{bI} and using the properties (B3) and (B5) , we get

$$\Pi^a_I \;\approx\; \frac{\alpha\sqrt{q}}{2}\overline{\lambda}\gamma_5\left(V^a_{[I}N_{J]}\right)\gamma^J\lambda \;=\; \frac{\alpha}{2}\overline{\lambda}\gamma_5\Pi^a_{IJ}\gamma^J\lambda. \tag{48}$$

This equation relates the torsion degrees of freedom encoded in Π^a_I with the spin degrees of freedom λ and $\overline{\lambda}$. Note that we have used only constraint equations and not equations of motion in deriving Eq.(48). This is a weak relation which comes naturally as a condition for the closure of the constraint algebra, since it has been derived by using the secondary constraint χ^{ab}. When there is no matter, this equation would indicate that torsion is zero [21]. Note that relation (48) is same as the one obtained in [15].

F. Closure of Constraint Algebra

At this point let us see what are the remaining requirements for obtaining a consistent Hamiltonian system. Recall that we had started with a constraint algebra with non-zero terms given in Eqs. (25-30). However, instead of Poisson Brackets we used Dirac Brackets constructed by using Ψ and $\overline{\Psi}$. Since H and Φ^a_{IJ} commutes with Ψ and $\overline{\Psi}$, the constraint algebra terms (25-27) remain the same. Also, now that $\Psi=0$ and $\overline{\Psi}=0$ are strong equalities, we do not need to worry about terms (28-30). Ensuring that the RHS of Eqs. (25-26) are weakly zero gave us a new constraint χ^{ab} and fixed the Lagrangian multiplier γ^I_a. The new secondary constraint χ^{ab} turns out to be second class, and we have additional non-zero terms in the constraint algebra as

$$\begin{aligned}
\left\{\Phi^c_{IJ}(\lambda^{IJ}_c),\chi^{ab}(\sigma_{ab})\right\} &= 2\sigma_{ab}\lambda^{IJ}_c\epsilon^{acd}\epsilon_{IJKL}V^b_K V^L_d, &(49)\\
\left\{\chi^{ab}(\sigma_{ab}),\Psi(u)\right\} &= 2iu\sigma_{ab}V^a_I V^{bI}N_J\gamma^J\lambda, &(50)\\
\left\{\chi^{ab}(\sigma_{ab}),\overline{\Psi}(\overline{u}\right\} &= -2i\overline{u}\sigma_{ab}V^a_I V^{bI}N_J\overline{\lambda}\gamma^J, &(51)\\
\left\{\chi^{ab}(\sigma_{ab}),C^c_I(\gamma^I_c)\right\} &= \frac{\alpha\sigma_{ac}}{2\sqrt{q}}\epsilon^{cdb}\gamma^{[I}_d V^{J]}_b\left(\Pi^a_{[I}N_{J]}-\alpha\Pi^a_{IJ}\overline{\lambda}\gamma_5 N_K\gamma^K\lambda\right) &(52)\\
&\quad+\frac{\sigma_{ab}}{\sqrt{q}}\gamma^I_c N_I\left(2\Pi^a_K V^c_J\Pi^b_{KJ}-\frac{\alpha}{2}\Pi^a_{JL}V^c_K\Pi^b_{JL}\overline{\lambda}\gamma_5\gamma^K\lambda\right)-\frac{2\alpha\sigma_{cb}}{\sqrt{q}}\epsilon^{cad}N_J\Pi^b_{IJ}D_a\gamma^I_d,\\
\left\{\chi^{ab}(\sigma_{ab}),H(N)\right\} &= \frac{\sigma_{ac}}{\sqrt{q}}\left[\Pi^a_{[I}N_{J]}-\alpha\Pi^a_{IJ}\overline{\lambda}\gamma_5 N_M\gamma^M\lambda\right]\left[D_b\left(\frac{N}{\sqrt{q}}\Pi^b_{[J|L|}\Pi^c_{I]L}\right)-\frac{N}{2}N_{[I}\Pi^c_{J]}\right. &(53)\\
&+\left.\frac{N\Pi^c_{KL}}{2\sqrt{q}}\left(\overline{\lambda}\sigma^{IJ}\sigma^{KL}\Pi+\overline{\Pi}\sigma^{KL}\sigma^{IJ}\lambda\right)\right]-\frac{2N}{q}\sigma_{cb}N_J\Pi^b_{IJ}\Pi^c_{IK}D_a\Pi^a_K\\
&+\frac{N\alpha\sigma_{ab}}{2q}\Pi^a_{IJ}\Pi^b_{IJ}\Pi^c_{ML}N_K\left(\overline{\lambda}\gamma_5\gamma^K\sigma^{ML}D_c\lambda-\overline{D_c\lambda}\sigma^{ML}\gamma_5\gamma^K\lambda\right)\\
&+\frac{\sigma_{ab}}{2\sqrt{q}}\left(4\Pi^a_K N_I V^c_J\Pi^b_{KJ}-\alpha\Pi^a_{JL}\Pi^b_{JL}V^c_K N_I\overline{\lambda}\gamma_5\gamma^K\lambda\right)D_c\left(NN^I\right).
\end{aligned}$$

Again, since we are using Dirac Brackets we do not consider Eqs. (50) and (51). Also, as mentioned above, the Poisson Brackets are weakly equal to the Dirac Brackets. The RHS of remaining equations have to be made weakly zero. The subsequent calculation is long and complicated. Here we only describe the procedure and more details are provided in the Appendix (C).

Since χ^{ab} is a secondary second-class constraint, we do not add it to the total Hamiltonian H_T which is still given by Eq. (32). Now in Eq. (32), N, N^a, Λ^{IJ} and λ^{IJ}_a are Lagrange multipliers which can be arbitrary functions ,but γ^I_a is fixed to the specific form (46) to ensure $\dot{\Phi}^a_{IJ}\approx 0$. Now we need to ensure

$$\begin{aligned}
\dot{C}^a_I(\eta^I_a) &= \left\{C^a_I(\eta^I_a),H_T\right\}\approx\left\{C^a_I(\eta^I_a),\left(\Phi^a_{IJ}(\lambda^{IJ}_a)+H(N)\right)\right\}\approx 0, &(54)\\
\dot{\chi}^{ab}(\sigma_{ab}) &= \left\{\chi^{ab}(\sigma_{ab}),H_T\right\}\approx\left\{\chi^{ab}(\sigma_{ab}),\left(\Phi^a_{IJ}(\lambda^{IJ}_a)+C^a_I(\gamma^I_a)+H(N)\right)\right\}\approx 0, &(55)
\end{aligned}$$

where η^I_a and σ_{ab} are arbitrary smearing functions. We solve the 18 independent equations (54) and (55) to fix the 18 independent components of the Lagrangian multiplier λ^{IJ}_a (see Appendix (C)). Note that the constraint H is still second class. However it can be shown that we can construct a first-class Hamiltonian constraint by

$$\tilde{H}=H+\frac{\gamma^I_a}{N}C^a_I+\frac{\lambda^{IJ}_a}{N}\Phi^a_{IJ}. \tag{56}$$

This will go back to H once we solve the second-class constraints in the next section.

It can be explicitly checked that the constraint algebra is now closed. We have obtained a consistent Hamiltonian system. The Lagrangian multipliers of the second-class constraints C^a_I and Φ^a_{IJ} have been fixed by the conditions of closure of the constraint algebra, while those of the first-class constraints $\mathcal{G}_{IJ}$, $\tilde{H}_a$ and $\tilde{H}$ remain free.

IV. SOLVING THE SECOND-CLASS CONSTRAINTS

Having obtained a consistent Hamiltonian system we now proceed to solve all the second-class constraints C^a_I, Φ^a_{IJ} and χ^{ab}. We will do this after performing a partial gauge fixing. Since we have already proved the consistency of the Hamiltonian system, we can be sure that making a gauge choice now will not lead to any inconsistency. Our goal is to reduce the internal $SO(3,1)$ gauge symmetry to $SU(2)$. So we break the $SO(3,1)$ by fixing the internal vector $N^I = (1,0,0,0)$, i.e.,we fix a specific timelike direction in the internal space. This is a standard gauge choice and is known as *time gauge*. From Eqs. (B1), (B2) and (B3) it is easy to see that

$$N^I = (1,0,0,0) \quad \Leftrightarrow \quad V^0_a \;=\; 0 \;=\; V^a_0. \tag{57}$$

For consistency, this gauge fixing condition has to be preserved, i.e.,

$$\dot{V}^0_a = \left\{V^0_a, H_T\right\}^\star \approx \left\{V^0_a, H_T\right\} \approx 0$$

Hence in time gauge we get

$$\Lambda^{0i} V_{ai} \;\approx\; \partial_a N. \tag{58}$$

The Lagrangian multiplier Λ^{0i} of $\mathcal{G}_{0i}$ gets fixed. This is expected because, by fixing N^I, we have broken the $SO(3,1)$ gauge invariance. The preservation of this gauge fixing condition implies that the boost part of the Gaussian constraint does not generate gauge transformations.

We first solve the constraint (19), which can be written as

$$\Pi^a_{IJ} = \frac{1}{2}\epsilon^{abc}\epsilon_{IJKL}V^K_b V^L_c.$$

Thus in time gauge we have

$$\Pi^a_{ij} = 0 \quad ; \quad \Pi^a_{0i} \;=\; \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}V^j_b V^k_c := E^a_i. \tag{59}$$

So, after solving this constraint only the Π^a_{0i} part of Π^a_{IJ} remains a dynamical variable. Consequently, only the ${\omega_a}^{0i}$ part of the $SO(3,1)$ connection remains dynamical.

For convenience we define $K^i_a := 2{\omega_a}^{0i}$ which will be conjugate to E^a_i. The ${\omega_a}^{ij}$ is the remaining part of the connection which will get solved in terms of other variables while solving the remaining constraints. Since our gauge group is now reduced to $SU(2)$, we will expand $SO(3,1)$ connection components ${\omega_a}^{ij}$ in the adjoint basis of $SU(2)$ as ${\omega_a}^{ij} := -\epsilon^{ijk}\Gamma_{ak}$. The quantity Γ_{ak} is the $SU(2)$ connection. Note that we had started with a $SO(3,1)$ spin connection ${\omega_a}^{ij}$ which was not torsion-free. Therefore the variables K^i_a and Γ^i_a that we define above will contain information about torsion implicitly. Also, from (B3) it is clear that, in time gauge, V^a_i is the inverse of V^i_a. Using the properties of inverses and determinants of matrices, it is easy to see from (59) that $E^a_i = \sqrt{q}V^a_i$ is the densitized triad. We can also determine its inverse $E^i_a = \frac{1}{\sqrt{q}}V^i_a$. [2]

Next we consider the constraint (40). Actually, we have already found the solution of this constraint by Eq. (48). It can be verified that the relation (48) identically solves the constraint. Using the above solution (59), in time gauge the solution (48) simplifies to

$$\Pi^a_0 = \frac{\alpha}{2}E^a_i\overline{\lambda}\gamma_5\gamma^i\lambda \quad ; \quad \Pi^a_i = -\frac{\alpha}{2}E^a_i\overline{\lambda}\gamma_5\gamma^0\lambda. \tag{60}$$

The torsion degrees of freedom are solved in terms of the densitized triad E^a_i and the fermionic fields λ and $\overline{\lambda}$.

Using above results we can now solve the remaining second-class constraint (18) as

$$C^a_0 = 0 \quad \Rightarrow \quad \frac{E^a_i}{2}\left(\overline{\lambda}\gamma_5\gamma^i\lambda - \frac{1}{\sqrt{q}}\epsilon_{ijk}K^j_b E^{bk}\right) \;=\; 0, \tag{61}$$

$$C^a_i = 0 \quad \Rightarrow \quad \frac{E^a_i}{2}\overline{\lambda}\gamma_5\gamma^0\lambda + \frac{1}{\sqrt{q}}\left(\epsilon^{jkl}E^a_j E^b_k E^c_l \partial_b E^i_c + \frac{1}{2}\epsilon^{ijk}E^a_j E^b_k E^l_c \partial_b E^c_l\right) + \frac{1}{\sqrt{q}}\left(\Gamma^k_b E^b_k E^a_i - \Gamma^k_b E^b_i E^a_k\right) \;=\; 0. \tag{62}$$

[2] In this work, we are not interested in the behaviour under parity transformations. Therefore we omit the sgn(detE^a_i) terms from our expressions.

Equation (62) can be used to solve the spin connection Γ_a^i in terms of the other variables. After some algebra we obtain

$$\begin{aligned}
\Gamma_a^i &= \frac{1}{2}\epsilon^{ijk}E_a^j E_k^b E_l^c \partial_b E_c^l + \frac{1}{2}\epsilon^{ijk}E_a^l E_j^b E_k^c \partial_b E_c^l + \frac{1}{2}\epsilon^{ijk}E_k^b \partial_a E_b^j - \frac{1}{2}\epsilon^{ijk}E_k^b \partial_b E_a^j - \frac{\sqrt{q}}{4}E_a^i \overline{\lambda}\gamma_5\gamma^0\lambda && (63)\\
&:= \widehat{\Gamma}_a^i - \frac{\sqrt{q}}{4}E_a^i\overline{\lambda}\gamma_5\gamma^0\lambda, && (64)
\end{aligned}$$

where we have denoted by $\widehat{\Gamma}_a^i$, the first four terms of (63) which do not depend on the fermions. It turns out that $\widehat{\Gamma}_a^i$ is exactly the $SU(2)$ spin connection which we would have obtained had there been no fermionic matter [5]. So, when there is no matter we go back to the standard formulation. Also note that the spin connection $\widehat{\Gamma}_a^i$ is independent of the arbitrary coupling parameter α.

Let us now consider the Gaussian constraint (17). It is obvious that because of the identity (42), the Gaussian constraint is given by

$$\mathcal{G}_{IJ} = D_a\Pi_{IJ}^a + \frac{\sqrt{q}}{4}\left(\epsilon_{IJKL}\overline{\lambda}\gamma_5 N^{[K}\gamma^{L]}\lambda\right) \approx 0. \tag{65}$$

In time gauge and using the solutions of Φ_{IJ}^a we can rewrite this as

$$\begin{aligned}
\mathcal{G}_{0i} &= \partial_a E_i^a + {\omega_a}^{ij}E_j^a = \partial_a E_i^a + \epsilon_{ijk}\Gamma_a^j E^{ak} \approx 0, && (66)\\
\mathcal{G}_{jk} &= \frac{1}{2}K_{b[j}E_{k]}^b - \frac{\sqrt{q}}{2}\epsilon_{jkl}\overline{\lambda}\gamma_5\gamma^l\lambda \approx 0. && (67)
\end{aligned}$$

On rewriting ${\omega_a}^{ij}$ in terms of Γ_a^i and inserting the solution (63), the $\mathcal{G}_{0i}$ term becomes identically zero, i.e., $D_aE_i^a = 0$. The boost part of the Gaussian constraint, which has no physical meaning after choosing time gauge, is therefore eliminated from our theory. Also comparing Eq. (67) with Eq. (61) it is easy to see that if $\epsilon_{jkl}\mathcal{G}_{jk} \approx 0$ then C_0^a is automatically zero (assuming $E_i^a \neq 0$).

So far, we have *reduced* our original phase space by consistently imposing time gauge and then solving all the second-class constraints. As a result, some basic variables in the original phase space have been eliminated in terms of the others. To obtain the basic variables in this phase space we need to study the $p\dot{q}$ term in Eq. (13),

$$\Pi_{IJ}^a \partial_t {\omega_a}^{IJ} + \Pi_I^a \partial_t V_a^I + \overline{\Pi}\partial_t\lambda + \Pi\partial_t\overline{\lambda}. \tag{68}$$

Using the solution (59) the first term becomes $E_i^a\partial_t K_a^i$. Using the solution (60) and the fact the preservation of time gauge requires that $\partial_t V_a^0 = 0$, the second term becomes $-\frac{\alpha}{2}\lambda^\dagger\gamma_5\lambda E_i^a\partial_t V_a^i$. For the last two terms in (68), recall that $\overline{\lambda} := \lambda^\dagger\gamma^0$. Also Π and $\overline{\Pi}$ can be read off from the identities (20). Then in time gauge, using the properties of the γ matrices given in appendix (A) we get

$$\overline{\Pi}\partial_t\lambda + \Pi\partial_t\overline{\lambda} = -i\sqrt{q}\lambda^\dagger\left(\partial_t\lambda\right) + i\sqrt{q}\left(\partial_t\lambda^\dagger\right)\lambda + \frac{\alpha}{2}\lambda^\dagger\gamma_5\lambda\ \partial_t\left(\sqrt{q}\right) - \partial_t\left(\frac{\alpha}{2}\sqrt{q}\lambda^\dagger\gamma_5\lambda\right).$$

Now since

$$\partial_t\sqrt{q} = \sqrt{q}V_i^a\partial_t V_a^i = E_i^a\partial_t V_a^i,$$

we have

$$\overline{\Pi}\partial_t\lambda + \Pi\partial_t\overline{\lambda} = -i\sqrt{q}\lambda^\dagger\left(\partial_t\lambda\right) + i\sqrt{q}\left(\partial_t\lambda^\dagger\right)\lambda + \frac{\alpha}{2}\lambda^\dagger\gamma_5\lambda E_i^a\partial_t V_a^i, \tag{69}$$

where we have neglected the total time derivative term. Putting everything together, (68) becomes

$$\begin{aligned}
& E_i^a\partial_t K_a^i - i\sqrt{q}\lambda^\dagger\left(\partial_t\lambda\right) + i\sqrt{q}\left(\partial_t\lambda^\dagger\right)\lambda \\
=\ & E_i^a\partial_t K_a^i + \left(\partial_t\zeta^\dagger\right)i\zeta - i\zeta^\dagger\left(\partial_t\zeta\right) \\
\equiv\ & E_i^a\partial_t K_a^i + \left(\partial_t\zeta^\dagger\right)\Pi_{\zeta^\dagger} + \Pi_\zeta\left(\partial_t\zeta\right), && (70)
\end{aligned}$$

where, following [19, 24], we have defined half-densities of the fermionic variables: $\zeta := \sqrt[4]{q}\lambda$ and $\zeta^\dagger := \sqrt[4]{q}\lambda^\dagger$. In the third line we have identified $\Pi_\zeta = -i\zeta^\dagger$ and $\Pi_{\zeta^\dagger} = i\zeta$. The basic variables of our system after solving all the second-class constraints are clearly shown in the expression (70). By using the half-densities, we have removed the $\sqrt{q}$ factors from the fermionic terms. Consequently, the fermionic sector and the gravitational sector now commute

with each other trivially. Therefore the Dirac brackets can only differ from the Poisson Brackets for the fermionic variables. The constraints Ψ and $\overline{\Psi}$ which we had used to construct the Dirac Brackets now become

$$\psi \ := \ \Pi_\zeta + i\zeta^\dagger \ \approx 0 \qquad ; \qquad \tilde{\psi} \ := \ \Pi_{\zeta^\dagger} - i\zeta \ \approx 0.$$

They satisfy

$$\left\{\psi, \tilde{\psi}\right\} = 2i. \tag{71}$$

So the determinant of the required matrix M_{mn} (see (31)) is not zero. Using this fact we can calculate the Dirac brackets for the fermion sector. The remaining 7 constraints, $\mathcal{G}_{jk}$, $\tilde{H}_a$ and H, are first class.

V. $SU(2)$ GAUGE THEORY

We have obtained a consistent Hamiltonian system which is invariant under local $SU(2)$ rotations. However this is not a $SU(2)$ gauge theory yet. The basic variables in the gravitational sector are the densitized triad E^a_i and its conjugate K^i_a. The connection Γ^i_a, given by Eq. (63) is a function of E^a_i, ζ and $\zeta^\dagger$. In this section, we shall make a canonical transformation to find a new variable conjugate to E^a_i and which also turns out to be a connection.

To do so, let us look at the Gaussian constraints (66) and (67) again. Recall that the first equation is identically zero while the second equation is a first-class constraint. We construct a new combination

$$\begin{aligned} G_i \ &:= \ \mathcal{G}_{0i} + \beta\epsilon_{ijk}\mathcal{G}^{jk} \\ &= \ \partial_a E^a_i + \epsilon_{ijk}\left(\Gamma^j_a + \beta K^j_a\right)E^{ak} - \beta\sqrt{q}\overline{\lambda}\gamma_5\gamma^i\lambda \ \approx \ 0 \quad , \end{aligned}$$

where β is any non-zero real number. Using Eq. (64) we get the standard $SU(2)$ Gaussian constraint with matter as

$$\begin{aligned} G_i \ &= \ \partial_a E^a_i + \epsilon_{ijk}\left(\widehat{\Gamma}^j_a + \beta K^j_a\right)E^{ak} - \beta\sqrt{q}\overline{\lambda}\gamma_5\gamma^i\lambda \\ &= \ \partial_a E^a_i + \epsilon_{ijk}A^j_a E^{ak} - \beta\sqrt{q}\overline{\lambda}\gamma_5\gamma^i\lambda \\ &\equiv \ \mathcal{D}_a E^a_i - \beta\zeta^\dagger\gamma^0\gamma_5\gamma_i\zeta \ \approx \ 0, \end{aligned} \tag{72}$$

where $A^i_a := \widehat{\Gamma}^i_a + \beta K^i_a$. Tensorially, the new connection A^i_a which we have defined is in the same form as the standard Ashtekar-Barbero connection without torsion. The basic variable Π^a_I which encoded the torsion has been solved in terms of the fermionic degrees of freedom when we solved the second-class constraint χ^{ab} in Eq. (60). We had started with a $SO(3,1)$ connection ${\omega_a}^{IJ}$ which is not torsion free. That fact is reflected in our expression of the $SU(2)$ spin connection Γ^i_a in Eq. (64). But in the new connection A^i_a which we define above, we remove exactly that additional piece. However, since we had defined $K^i_a := 2{\omega_a}^{0i}$, the variable K^i_a implicitly contains information about the torsion. When there is no matter, torsion goes to zero and the S_T term in our action (1), and therefore, the terms originating from it in the Hamiltonian analysis vanish [21]. Then we go back to the standard formalism with a torsion-free $SO(3,1)$ spin connection.

With our connection A^i_a, we can also easily prove the canonical commutation relations following [5] (with $\kappa = 1$) as

$$\begin{aligned} \left\{E^a_i(x), A^j_b(y)\right\} \quad &= \quad \delta^a_b\delta^j_i\delta(x,y), \\ \left\{E^a_i(x), E^b_j(y)\right\} \quad = \quad 0 \quad &= \quad \left\{A^i_a(x), A^j_b(y)\right\}. \end{aligned} \tag{73}$$

We have obtained a $SU(2)$ gauge theory formulation of our system. A peculiar result of our analysis is that, the coupling constant α of S_T in the action (1) is absent from the definition of A^i_a. It depends on an arbitrary number β which may not have any relation with the parameter α with which we started. However, in the analysis performed with Holst term or the Nieh-Yan term , the connection depends on the arbitrary coupling parameter γ present in the action.

The remaining constraints can also be rewritten in terms of the new basic variables. Using $K^i_a = \frac{1}{\beta}(A^i_a - \widehat{\Gamma}^i_a)$ and the Gaussian constraint (72), the diffeomorphism constraint (15) can be written as

$$\begin{aligned} H_a \ &= \ E^b_i\partial_{[a}K^i_{b]} - K^i_a\partial_b E^b_i + \frac{1}{2}\bigg(\pi_\zeta\partial_a\zeta - (\partial_a\pi_\zeta)\zeta + (\partial_a\zeta^\dagger)\pi_{\zeta^\dagger} - \zeta^\dagger\partial_a\pi_{\zeta^\dagger}\bigg) \\ &\approx \ \frac{1}{\beta}E^b_i F^i_{ab} - A^i_a\zeta^\dagger\gamma^0\gamma_5\gamma_i\zeta + \frac{1}{2}\bigg(\pi_\zeta\partial_a\zeta - (\partial_a\pi_\zeta)\zeta + (\partial_a\zeta^\dagger)\pi_{\zeta^\dagger} - \zeta^\dagger\partial_a\pi_{\zeta^\dagger}\bigg), \end{aligned} \tag{74}$$

where $F^i_{ab} := \partial_{[a}A^i_{b]} + \epsilon^i_{jk}A^j_a A^k_b$ is the curvature of A^i_a. This is exactly the standard diffeomorphism constraint. Note that, although the arbitrary parameter β appears, the α dependent terms have again dropped off from the final expression. The Hamiltonian constraint(14) is more complicated. After some calculation, we get

$$\begin{aligned} H \;=\; & \frac{1}{\sqrt{q}}\left[\epsilon_{ijk}E^a_iE^b_jF^k_{ab} - \left(\beta^2+\frac{1}{4}\right)E^a_iE^b_jK^i_{[a}K^j_{b]}\right] + \frac{i}{4\sqrt{q}}\left(\pi_\zeta\gamma_5\zeta - \zeta^\dagger\gamma_5\pi_{\zeta^\dagger}\right)\epsilon_{ijk}E^a_iE^b_j\partial_aE^k_b \\ & -\frac{9}{32\sqrt{q}}\left(\pi_\zeta\gamma_5\zeta - \zeta^\dagger\gamma_5\pi_{\zeta^\dagger}\right)\left(\pi_\zeta\gamma_5\zeta - \zeta^\dagger\gamma_5\pi_{\zeta^\dagger}\right) - \frac{1}{2\sqrt{q}}\left(\pi_\zeta\tau_i\zeta - \zeta^\dagger\tau_i\pi_{\zeta^\dagger}\right)\left(\pi_\zeta\tau_i\zeta - \zeta^\dagger\tau_i\pi_{\zeta^\dagger}\right) \\ & +2\beta\partial_a\left[\frac{1}{\sqrt{q}}E^a_i(\pi_\zeta\tau_i\zeta - \zeta^\dagger\tau_i\pi_{\zeta^\dagger})\right] + \frac{2E^a_i}{\sqrt{q}}\left((\partial_a\zeta^\dagger)\sigma^{0i}\pi_{\zeta^\dagger} + \pi_\zeta\sigma^{0i}\partial_a\zeta\right), \end{aligned} \tag{75}$$

where $\tau_i = -\frac{i}{2}\sigma_i$ and σ_i are Pauli matrices. Again this expression is independent of α. This expression goes over to the standard expression when the fermions are set to zero. Thus we complete our task of obtaining a $SU(2)$ gauge theory.

VI. CONCLUSION

Let us briefly summarize what we have achieved in this paper. We started with an action containing a torsion-squared term and fermionic matter apart from the standard Hilbert-Palatini term. This T^2 term is just the difference between the total derivative Nieh-Yan term and the Holst term. Since an $SU(2)$ gauge theory formulation can be derived from actions containing either [12, 15], it seemed possible that such a formulation can also be obtained from our action containing only the T^2 term. We also need to add fermionic matter because the vacuum case is torsion free [21] and we are left with only the Hilbert-Palatini part which does not admit a gauge theory formulation. We take non-minimally coupled fermionic matter so that the classical equations of motion for the fermions do not depend on the coupling constant α multiplying the torsion term.

We do a $3+1$ decomposition of our action, do a constraint analysis and finally obtain a consistent Hamiltonian system with second-class constraints. The second-class constraints related to the definition of fermion momentum are treated in the standard way via Dirac Brackets, while the remaining second-class constraints are solved after breaking the $SO(3,1)$ invariance by fixing time gauge. As far as we know, such Hamiltonian analysis on an action with non-zero torsion term with explicit expressions of all the second-class constraints is new in literature. Similar analysis with the Holst term (with non-minimally coupled fermions) [17–19] and the Nieh-Yan term (with minimally coupled fermions) [15] has already been attempted before, although the picture is not yet complete. Apart from the crucial fact that the gravitational part of our action being different from those studied in literature so far, there are several other differences in our approach. Since we are motivated by PGT where the initial action is invariant under Poincare transformations, our starting variables are different from those used in [17, 18]. Unlike the treatment in [19] we do not break up our variables into the torsion dependent and independent pieces. Moreover, since we do not have the Holst term, the techniques developed in [22] for dealing with second-class constraints and used in [15] are not available to us. Also, unlike the treatment in [17–19], we fix the time gauge only after we have found all the second-class constraints and obtained a consistent Hamiltonian system. Also, in [19], there are two different coupling parameters, one for the Holst term γ which is different in general from the parameter α in the non-minimal fermion action. In our case the parameter in the T^2 term gets related to the one in the fermion action due to the closure of constraint algebra.

On solving the second-class constraint χ^{ab}, torsion gets related to the fermionic degrees of freedom via Eq.(48) which is as same as the one obtained in [15]. Further, solution of the second-class constraint C^a_i gives the $SU(2)$ spin connection Γ^i_a in terms of the densitized triad. This differs from the spin connection in GR [5] only by a term which depends on the fermions. In the final step we obtain the connection dynamics by defining a new connection A^i_a which is algebraically in the same form as the Ashtekar-Barbero connection without torsion. However, unlike the torsion-free case, the K^i_a part comes from the ${\omega_a}^{0i}$ part of the $SO(3,1)$ connection which is not torsion free. As a result it is not obvious the K^i_a can be directly related to the extrinsic curvature K_{ab} on shell. While the diffeomorphism constraint (74) is standard, the additional terms in our Hamiltonian constraint (75) are somehow different from the ones obtained in literature. Although these constraints can be loop quantized using existing techniques, it may be possible to rewrite them in a form more convenient for loop quantization. We leave this issue for future research. This present work at least opens the door to extending loop quantization techniques from standard GR to more general PGT of gravity.

One more peculiar and novel feature of our analysis is that the coupling constant α in our starting action totally disappears from the final Hamiltonian system. Rather the new connection A^i_a depends on an arbitrary new parameter

β which does not have any relation with α. This is unlike all the analysis done with Holst and Nieh-Yan terms where the Immirzi parameter γ in the definition of the Ashtekar-Barbero connection is just the coupling parameter in the action (up to numerical factors). The nature and the role of this new parameter is not clear so far and deserve investigating to other actions with torsion terms.

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Appendix A: Gamma Matrix

In this section we collect some of the standard properties of Dirac matrices which we have used in previous sections. The γ matrices, in any dimension, satisfy the Clifford algebra

$$\{\gamma_I, \gamma_J\} = \gamma_I\gamma_J + \gamma_J\gamma_I = 2\eta_{IJ} \tag{A1}$$

where η_{IJ} is the flat Minkowski metric. We shall restrict ourselves to 4 dimensions and choose the signature $(-+++)$ which is different from the signature usually used in QFT. In this signature the above relation can be decomposed as

$${\gamma_0}^2 = -\mathbb{I}_4 \qquad ; \qquad {\gamma_i}^2 = \mathbb{I}_4.$$

This implies that γ_0 is anti-Hermitian while γ_i is Hermitian. Note that all the γ matrices are unitary. We also define the commutator $\sigma_{IJ} := \frac{1}{4}[\gamma_I, \gamma_J]$ and another standard combination $\gamma_5 := i\gamma_0\gamma_1\gamma_2\gamma_3$. It is easy to check that $(\gamma_5)^2 = \mathbb{I}$ and $(\gamma_5)^\dagger = \gamma_5$. In the Weyl representation, commonly used for massless fermions, the Dirac matrices can be explicitly written as

$$\gamma_0 = \begin{pmatrix} 0 & i\mathbb{I}_2 \\ i\mathbb{I}_2 & 0 \end{pmatrix} \qquad ; \qquad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \qquad ; \qquad \gamma_5 = \begin{pmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}.$$

In this paper we have used the following standard identities

$$\begin{aligned} \{\gamma_5, \gamma_I\} \quad = \quad 0 \quad = \quad [\gamma_5, \sigma_{IJ}]\,; \\ [\gamma_K, \sigma_{IJ}] = \eta_{K[I}\gamma_{J]} \qquad ; \qquad \{\gamma_K, \sigma_{IJ}\} = i\epsilon^{KIJL}\gamma_5\gamma_L. \end{aligned} \tag{A2}$$

Appendix B: $3+1$ Decomposition

In this section we give the parametrization of the tetrad and the co-tetrad fields which we use in this paper. They read

$$\begin{aligned} e_{tI} = NN_I + N^aV_{aI} \qquad &; \qquad e^{tI} = -\frac{N^I}{N}, \\ e_{aI} = V_{aI} \qquad &; \qquad e^{aI} = V^{aI} + \frac{N^aN^I}{N}, \end{aligned} \tag{B1}$$

$$\text{with} \qquad N^IV_{aI} = 0 \qquad ; \qquad N^IN_I = -1. \tag{B2}$$

What we have done is that we have reparametrized the 16 degrees of freedom of $e_{\mu I}$ into 20 fields given by (B1) subject to the 4 constraints (B2). From these definitions, the following identities also hold:

$$\begin{aligned} & V^{aI}V_{bI} = \delta^a_b \quad ; \quad V^{aI}N_I = 0 \quad ; \quad N_a := V_{aI}V^I_bN^b, \\ & V^{aI}{V_a}^J = \eta^{IJ} + N^IN^J. \end{aligned} \tag{B3}$$

In terms of these fields the metric takes the standard form

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^aN_a & N_a \\ N_a & V_{aI}V^I_b \end{pmatrix}.$$

It is easy to see that

$$
\begin{aligned}
g &:= \det(g_{\mu\nu}) = -N^2 \det(V_{aI} V_b^I), \\
e &:= |\det(e_{\mu I})| = N\sqrt{\det(V_{aI} V_b^I)} = N\sqrt{\det(q_{ab})} = N\sqrt{q}.
\end{aligned}
$$

Using the definitions given above we can also prove the following two identities which have been used in our analysis,

$$
\begin{aligned}
-\,e e^a_{[I} e^b_{J]} &= \frac{N^2}{e} \Pi^{[a}_{IK} \Pi^{b]}_{JL} \eta^{KL} + N^{[a} \Pi^{b]}_{IJ}, && \text{(B4)} \\
\Pi^a{}_{IJ} &= \sqrt{q} V^a_{[I} N_{J]} \;\;\Rightarrow\;\; V^a_I = -\frac{1}{\sqrt{q}} \Pi^a{}_{IJ} N^J. && \text{(B5)}
\end{aligned}
$$

Appendix C: Determination of λ_a^{IJ}

In this section we show how to obtain λ_a^{IJ} from Eqs. (54) and (55). First let us consider $\dot{C}^a_I$. Using Eqs. (26) and (27), we get

$$
\begin{aligned}
\dot{C}^a_I(\eta^I_a) &= \left\{ C^a_I(\eta^I_a), \left(\Phi^a_{IJ}(\lambda^{IJ}_a) + H(N) \right) \right\} \\
&= \epsilon^{abc} \eta^I_b V^J_c \left[\left(\alpha \lambda^{IJ}_a + \epsilon_{IJKL} \lambda^{KL}_a \right) - \frac{\alpha N}{\sqrt{q}} \left(\overline{D_a \lambda}\, \sigma_{IJ} \Pi - \overline{\Pi}\, \sigma_{IJ} D_a \lambda \right) \right] \quad , \quad \forall\, \eta^I_b. \qquad \text{(C1)}
\end{aligned}
$$

For convenience, we define

$$
X^{IJ}_a := \left(\alpha \lambda^{IJ}_a + \epsilon_{IJKL} \lambda^{KL}_a \right) - \frac{\alpha N}{\sqrt{q}} \left(\overline{D_a \lambda}\, \sigma_{IJ} \Pi - \overline{\Pi}\, \sigma_{IJ} D_a \lambda \right). \qquad \text{(C2)}
$$

This is the quantity we shall solve for. The equation (C2) can easily be inverted to express λ^{IJ}_a in terms of X^{IJ}_a as

$$
\lambda^{IJ}_a = \frac{\alpha}{\alpha^2 + 4} \left[\frac{\alpha N}{\sqrt{q}} \left(\overline{D_a \lambda}\, \sigma_{IJ} \Pi - \overline{\Pi}\, \sigma_{IJ} D_a \lambda \right) - \frac{N}{\sqrt{q}} \epsilon_{IJKL} \left(\overline{D_a \lambda}\, \sigma^{KL} \Pi - \overline{\Pi}\, \sigma^{KL} D_a \lambda \right) + X^{IJ}_a - \frac{1}{\alpha} \epsilon_{IJKL} X^{KL}_a \right]. \qquad \text{(C3)}
$$

Thanks to this definition, it is easy to see from (C1) that

$$
\dot{C}^a_I \;\approx 0 \;\Rightarrow\; \epsilon^{abc} V^J_c X^{IJ}_a = 0. \qquad \text{(C4)}
$$

Now let us consider $\dot{\chi}^{ab}$. We have

$$
\begin{aligned}
\dot{\chi}^{ab}(\sigma_{ab}) &= \left\{ \chi^{ab}(\sigma_{ab}), \Phi^a_{IJ}(\lambda^{IJ}_a) \right\} + \left\{ \chi^{ab}(\sigma_{ab}), \left(C^a_I(\gamma^I_a) + H(N) \right) \right\} \\
&= -2 \sigma_{ab} \lambda^{IJ}_c \epsilon^{acd} \epsilon_{IJKL} V^b_K V^L_d + \sigma_{ab} \Sigma^{ab} \quad , \quad \forall\, \sigma_{ab} \qquad \text{(C5)}
\end{aligned}
$$

where we have defined $\sigma_{ab}\Sigma^{ab} := \left\{ \chi^{ab}(\sigma_{ab}), \left(C^a_I(\gamma^I_a) + H(N) \right) \right\}$. The explicit form of Σ^{ab} is very complicated and can be calculated using Eqs. (52) and (53). However we do not need the explicit form of Σ^{ab}. We are interested in solving for λ^{IJ}_a which only comes from the first part of Eq. (C5). After some more algebra we obtain the equation in terms of X^{IJ}_a as

$$
\begin{aligned}
\dot{\chi}^{cd} \;\approx\; 0 \;\Rightarrow\; \quad & \Sigma^{cd} + \frac{\alpha}{\alpha^2 + 4} \Big[2 V^c_I V^d_I \Pi^a_{KL} X^{KL}_a - \left(V^a_I V^d_I \Pi^c_{KL} + V^a_I V^c_I \Pi^d_{KL} \right) X^{KL}_a \\
& - \epsilon^{cab} \left(\frac{\alpha N}{\sqrt{q}} \epsilon_{IJKL} \left(\overline{D_a \lambda}\, \sigma^{KL} \Pi - \overline{\Pi}\, \sigma^{KL} D_a \lambda \right) + \frac{4N}{\sqrt{q}} \left(\overline{D_a \lambda}\, \sigma^{IJ} \Pi - \overline{\Pi}\, \sigma^{IJ} D_a \lambda \right) \right) V^d_I V^J_b \\
& - \epsilon^{dab} \left(\frac{\alpha N}{\sqrt{q}} \epsilon_{IJKL} \left(\overline{D_a \lambda}\, \sigma^{KL} \Pi - \overline{\Pi}\, \sigma^{KL} D_a \lambda \right) + \frac{4N}{\sqrt{q}} \left(\overline{D_a \lambda}\, \sigma^{IJ} \Pi - \overline{\Pi}\, \sigma^{IJ} D_a \lambda \right) \right) V^c_I V^J_b \Big] \\
& \approx\; 0. \qquad \text{(C6)}
\end{aligned}
$$

Using (C4) and (C6), after a long calculation we get

$$
\begin{aligned}
X_a^{IJ} \;=\; & \frac{1}{4\sqrt{q}}\left[V_{a[I}N_{J]}\epsilon^{def}A_d^{KL}V_{eK}V_{fL}+V_{c[I}N_{J]}\epsilon^{cef}A_e^{KL}V_{aK}V_{fL}+A_{eK[J}N_{I]}\epsilon^{def}V_{dL}V_a^{L}V_{fK}\right] \\
& + \frac{\alpha^2+4}{4\alpha\sqrt{q}}\left[\frac{1}{2}V_{a[I}N_{J]}\Sigma^{cd}V_c^{K}V_{dK}-V_{c[I}N_{J]}\Sigma^{cd}V_a^{K}V_{dK}\right] \qquad \text{(C7)}
\end{aligned}
$$

where, for brevity of notation, we have defined

$$
A_a^{IJ} := \frac{\alpha N}{\sqrt{q}}\epsilon_{IJKL}\left(\overline{D_a\lambda}\;\sigma^{KL}\Pi-\overline{\Pi}\;\sigma^{KL}D_a\lambda\right)+\frac{4N}{\sqrt{q}}\left(\overline{D_a\lambda}\;\sigma^{IJ}\Pi-\overline{\Pi}\;\sigma^{IJ}D_a\lambda\right).
$$

Putting (C7) into (C3), we get λ_a^{IJ}.

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